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Two hidden symmetries of the equations of ideal gasdynamics, and the general solution in a case of non-uniform entropy distribution

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Through an extension of the concept of scale invariance we construct two invariance transformations of the equations of gasdynamics: the first is a true Bäcklund transformation, and as such we expect that it should play an important role in the search for new cases of integrability; the other one affects the shape of the entropy profile, implying that the mathematical problems of isentropic and non-isentropic flow are, at least in some cases, equivalent.

That property enables us to deduce the general solution in closed form, assuming a power-law entropy profile of the form $P/\rho^\gamma \approx 1/M^{3\gamma-1}$, where M is the Lagrangian mass coordinate.

1. Introduction

The equations of gasdynamics have the well-known property of being invariant under scale transformations, as a consequence of their homogeneity. This property has many important and useful consequences, which have already been extensively discussed in the literature (e.g. Sedov 1959), such as the existence of so-called self-similar solutions, or the fact that any given flow may be completely characterized – up to the arbitrariness in the choice of units – by a set of purely dimensionless field variables. We present an extension of this concept of scale invariance, in which each element of fluid is assumed to 'carry with it' its own system of units; this is a natural generalization to consider, since in the macroscopic picture each fluid element moves without ever mixing with the others. The resulting symmetry of the Euler equations, presented in §2, has new and interesting properties and leads us to the general solution of the equations in closed form, in a case characterized by a non-uniform entropy distribution (§4).

There is one more symmetry (§3.3) of the Euler equations that may be found through an extension of the concept of scale invariance; we derive it through the consideration of time-dependent systems of units, rather than space-dependent as above. That new symmetry does not alter the entropy distribution, and thus constitutes a Bäcklund transformation of the hydrodynamical equations. Its existence raises the hope of finding new cases of integrability; it also suggests that the search for Riemann invariants should play a crucial role for that purpose. We show in fact in §5 that the data of a single Riemann invariant is sufficient in order to obtain at least a formal solution of the equations, as a result of the existence of the Bäcklund transformation.

2. Microscopic description of the symmetry

2.1. *The case of gasdynamics, without external forces*

We consider here the one-dimensional motion of a fluid, which we do not *a priori* constrain to be ideal or polytropic. A mass element may be labelled by the Lagrangian coordinate M, which represents the total mass enclosed between that element and another arbitrarily chosen as origin. With obvious notations, we have the following differential expression for M:

$$\mathrm{d}M = \rho(\mathrm{d}r - v\,\mathrm{d}t). \tag{2.1}$$

The associated integrability condition is called the continuity equation:

$$\frac{\partial(\rho v)}{\partial r} + \frac{\partial(\rho)}{\partial t} = 0. \tag{2.2}$$

We now rescale the fundamental units of mass, length and time, through scaling factors μ', s, τ that are explicitly dependent upon the Lagrangian coordinate M:

$$\mathrm{d}\overline{M} = \mu'(M)\,\mathrm{d}M, \quad \frac{\partial \bar{r}}{\partial r} = s(M), \quad \frac{\mathrm{d}}{\mathrm{d}\bar{t}} = \frac{1}{\tau(M)}\frac{\mathrm{d}}{\mathrm{d}t}, \tag{2.3a, b, c}$$

where a bar denotes transformed quantities; this transformation will be denoted by ($\overline{\mathrm{T}}$). The symbol $\mathrm{d}/\mathrm{d}t$ has here its usual meaning of a derivative taken at constant M, so that the condition (2.3c) is equivalent to the statement that $\bar{t}(M,t)$ is a linear function of time.

In the simple case where τ is chosen to be constant, the following integrated formulae hold:

$$\overline{M} = \mu(M), \quad \bar{t} = \tau t, \quad \bar{r} = \int_{M=0}^{M} s(M)\,\mathrm{d}r + R(t). \tag{2.4a, b, c}$$

The transformed velocity $\bar{v} \equiv \mathrm{d}\bar{r}/\mathrm{d}\bar{t}$ and acceleration $\bar{g} \equiv \mathrm{d}\bar{v}/\mathrm{d}\bar{t}$ read

$$\tau\bar{v} = \int_0^M s(M)\,\mathrm{d}v + \dot{R}(t), \quad \tau^2\bar{g} = \int_0^M s(M)\,\mathrm{d}g + \ddot{R}(t) \tag{2.5a, b}$$

(where g is the acceleration field), as may be shown through use of the continuity equation; the transformed density is

$$\bar{\rho} = \frac{\mu'(M)\rho}{s(M)}. \tag{2.6}$$

The Euler equation of motion reads, in the new reference frame,

$$\frac{\partial \overline{P}}{\partial \overline{M}} = -\bar{g},$$

and serves to determine the transformed pressure $\overline{P}$:

$$\overline{P} = -\int_{P=0}^{P} \bar{g}\,\mathrm{d}\overline{M}, \tag{2.7}$$

or, taking account of the original Euler equation ($P = -\int_0^P g\,\mathrm{d}M$),

$$\overline{P} = \int_{P=0}^{P} \frac{\bar{g}}{g}\mu'(M)\,\mathrm{d}P, \tag{2.8}$$

where the integral is taken at fixed t.

The only condition needed for the above transformation to produce a physically acceptable flow is that the transformed variables $\bar{P}$, $\bar{\rho}$, $\bar{M}$ be compatible with the equation of state of a realistic fluid.

As a particularly interesting instance, we propose the solution

$$s(M) = -\tau^2 M, \quad \mu(M) = 1/M, \tag{2.9a,b}$$

from which we derive

$$\bar{P} = P/M, \quad \ddot{R}(t) = -\tau^2 P_0(t), \tag{2.10a,b}$$

where $P_0(t) \equiv P(M = 0, t)$.

Equation (2.9) ($\bar{M} = 1/M$) shows the symmetrical nature of ($\bar{\mathrm{T}}$), in the sense that the product of ($\bar{\mathrm{T}}$) by itself is the identity. In order to make it more readily apparent, we choose the normalization constant $\tau = -1$, which produces the following set of transformation formulae:

$$\left.\begin{aligned} \bar{t} = -t, \quad \bar{M} = \frac{1}{M}, \quad \bar{P} = \frac{P}{M}, \quad \bar{\rho} = \frac{\rho}{M^3}, \\ \bar{r} = -\int_{M=0}^{M} M\,\mathrm{d}r + R(t), \quad \bar{v} = +\int_{0}^{M} M\,\mathrm{d}v - \dot{R}(t), \end{aligned}\right\} \tag{2.11}$$

with

$$\ddot{R}(t) = -P_0(t).$$

The above solution has the essential property that the polytropic equation of state

$$\frac{\mathrm{d}\log P}{\mathrm{d}t} = \gamma \frac{\mathrm{d}\log \rho}{\mathrm{d}t} \tag{2.12}$$

remains invariant; therefore ($\bar{\mathrm{T}}$) is an invariance transformation (a symmetry) of the Euler equations of one-dimensional gasdynamics. It has the very remarkable and unique property of transforming a given distribution of entropy into another *of a different shape*; namely, the entropy being the logarithm of σ,

$$\sigma(M) \equiv P/\rho^{\gamma}, \tag{2.13}$$

we have the following transformation formula:

$$\bar{\sigma}(\bar{M}) = M^{3\gamma-1}\sigma(M). \tag{2.14}$$

In particular, an isentropic distribution (σ = constant) gives rise to a power-law entropy profile:

$$\sigma(M) \approx 1/M^b \tag{2.15}$$

with an 'entropy index' $b = 3\gamma - 1$. That fundamental property will enable us to deduce the general solution of the Euler equations in closed form (§4) for such entropy distributions:

$$\sigma(M) \approx 1/M^{3\gamma-1}. \tag{2.16}$$

2.2. *Microscopic description, and generalization*

It is essential to notice the following generalization. Considered from the microscopic level where the Lagrangian coordinate M is an absolute constant (the diffusion processes being here neglected), *the above-described transformation reduces to an ordinary time-independent change of the fundamental units, and therefore preserves the form of the fundamental laws of mechanics. Both coordinate systems are inertial*: the average accelerations g, $\bar{g}$ of a small bunch of particles are merely the result of collisions with particles belonging to the neighbouring elements, and have the value predicted by Newton's second law, since both Euler equations are satisfied.

Then, if other forces – such as viscosity, gravitation, etc. – were present, *the above transformation would still constitute an invariance transformation of the more complex system of equations* – except for a possible M-dependent rescaling of the extra forces, if their strength is determined by a dimensional constant, such as the gravitational constant G.

In the pure case of gasdynamics, without any forces other than pressure, no such dimensional constant occurs, and the transformation is thus an exact invariance transformation.

3. Characteristic equations, and conservation laws

3.1. *Characteristic form of the Euler equations*

The characteristic formulation of the equations is a basic tool in any analytical study in hydrodynamics; we need it in particular in §5, where we solve the equations by the method of Riemann invariants, which are a particular set of characteristic coordinates. We refer the reader to Courant & Friedrichs (1948) for the method of derivation, given the one-dimensional Euler equations

$$\frac{\partial v}{\partial r}+\frac{\mathrm{d}\log\rho}{\mathrm{d}t}=0,\quad \frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial P}{\rho\,\partial r}=0,\quad \frac{P}{\rho^{\gamma}}=\sigma(M),$$

and merely indicate the result (3.1 a,b,c)

$$\partial_\alpha r=(v-c)\,\partial_\alpha t,\quad \partial_\beta r=(v+c)\,\partial_\beta t, \tag{3.2a,b}$$

$$\partial_\alpha\left(v-\frac{2c}{\gamma-1}\right)=\frac{-c}{\gamma(\gamma-1)}\partial_\alpha\log\sigma,\quad \partial_\beta\left(v+\frac{2c}{\gamma-1}\right)=\frac{+c}{\gamma(\gamma-1)}\partial_\beta\log\sigma, \tag{3.3a,b}$$

$$\partial_\alpha\Psi=-c^{u}\,\partial_\alpha t,\quad \partial_\beta\Psi=+c^{u}\,\partial_\beta t, \tag{3.4a,b}$$

with
$$u=(\gamma+1)/(\gamma-1).$$

Here α, β are the characteristic coordinates, ∂_α and ∂_β denote the partial derivatives $\partial/\partial\alpha$ and $\partial/\partial\beta$, $c\equiv(\gamma P/\rho)^{\frac{1}{2}}$ is the sound velocity, and Ψ is a function of M, or of σ, defined as

$$\Psi=\int(\gamma\sigma)^{1/(\gamma-1)}\,\mathrm{d}M, \tag{3.5}$$

or, equivalently,

$$\frac{\partial\Psi}{\partial r}=c^{u-1},\quad \frac{\mathrm{d}\Psi}{\mathrm{d}t}=0.$$

It is worth noticing the following relation, involving the pressure P:

$$\Psi=\frac{\gamma-1}{\gamma(\gamma-1-b)}\frac{Mc^{2\gamma/(\gamma-1)}}{P}. \tag{3.6}$$

In terms of M instead of Ψ, the characteristic equations (3.4) read

$$\partial_\alpha M=-\rho c\,\partial_\alpha t,\quad \partial_\beta M=+\rho c\,\partial_\beta t.$$

We will sometimes refer to (3.2), (3.3) and (3.4) as the first, second and third group of characteristic equations respectively.

The condition (3.5) expresses that Ψ is a function of σ, whence the differential relation

$$\partial_\alpha\Psi\,\partial_\beta\sigma=\partial_\beta\Psi\,\partial_\alpha\sigma. \tag{3.7}$$

Equation (3.7) is a consequence of the system (3.2)–(3.4), which is thus of 5th order (with 5 unknowns: r, t, v, c, σ).

The original Euler equations (3.1) form a system of the 3rd order only; the increase in order (from 3rd to 5th) is ascribable to the introduction of characteristic coordinates, which are only defined up to a gauge transformation of the form

$$\alpha \to \alpha^* = \phi(\alpha), \quad \beta \to \beta^* = \psi(\beta), \tag{3.8a, b}$$

(where ϕ and ψ are arbitrary functions), as is easily seen from the form of the characteristic equations. This fundamental property will be again considered in §5.1.

The first group of equations has an obvious physical meaning: it defines the characteristic curves as the trajectories of 'small-amplitude perturbations', which are known to propagate at the velocity of sound c. We may also note that the second members of the three pairs of equations (I, II, III), i.e. those involving the partial derivative ∂_β, may be deduced from the other equations by exchanging α and β, and changing the signs of c and Ψ only.

3.2. *Conservation laws*

In one-dimensional problems, conservation laws assume the general form

$$\frac{\partial j_{\mathrm{E}}}{\partial r} + \frac{\partial \rho_{\mathrm{E}}}{\partial t} = 0,$$

where j_{E}, ρ_{E} are *a priori* arbitrary expressions defined in terms of physical variables, such as r, t, P, v, ρ, c; this formulation may be called Eulerian since the independent variables are r and t. Its form is that of a Cauchy integrability condition, of a quantity A defined by the pair of equations:

$$\frac{\partial A}{\partial t} = -j_{\mathrm{E}}, \quad \frac{\partial A}{\partial r} = +\rho_{\mathrm{E}}.$$

A may be called the conserved quantity, and j_{E}, ρ_{E} are its (Eulerian) current and density.

The most compact way to formulate a conservation law (for a conserved quantity A), however, is usually through the Lagrangian formalism: defining the 'Lagrangian current' as $j_{\mathrm{L}}[A] = -\mathrm{d}A/\mathrm{d}t$ and the Lagrangian density as $\rho_{\mathrm{L}}[A] = \partial A/\partial M$, the conservation law reads

$$\frac{\partial j_{\mathrm{L}}}{\partial M} + \frac{\mathrm{d}\rho_{\mathrm{L}}}{\mathrm{d}t} = 0. \tag{3.9}$$

Taking account of the continuity equation, (3.9) is shown to be equivalent to the usual Eulerian formulation:

$$\frac{\partial}{\partial r}(j_{\mathrm{L}} + \rho v \rho_{\mathrm{L}}) + \frac{\partial}{\partial t}(\rho \rho_{\mathrm{L}}) = 0. \tag{3.10}$$

The currents and densities of momentum Π and energy E thus read

$$\left.\begin{aligned} j_{\mathrm{L}}[\Pi] &= P, \quad \rho_{\mathrm{L}}[\Pi] = v, \\ j_{\mathrm{L}}[E] &= Pv, \quad \rho_{\mathrm{L}}(E) = \frac{v^2}{2} + \frac{c^2}{\gamma(\gamma-1)}. \end{aligned}\right\} \tag{3.11}$$

These expressions determine Π and E, up to an arbitrary additive constant.

The 'characteristic formulation' (of a conservation law) is most easily derived through the formulae

$$\frac{\mathrm{d}}{\mathrm{d}t} - \rho c \frac{\partial}{\partial M} = \frac{\partial_\alpha}{\partial_\alpha t}, \quad \frac{\mathrm{d}}{\mathrm{d}t} + \rho c \frac{\partial}{\partial M} = \frac{\partial_\beta}{\partial_\beta t}. \tag{3.12a, b}$$

We thus obtain the following expression for the differential of Π:

$$\partial_\alpha \Pi = \left(\frac{\mathrm{d}\Pi}{\mathrm{d}t} - \rho c \frac{\partial \Pi}{\partial M}\right) \partial_\alpha t = -(P + \rho c v)\,\partial_\alpha t,$$

and, taking account of the characteristic equation (3.4), written in the form $\partial_\alpha M = -\rho c\,\partial_\alpha t$, together with the identity $\gamma P = \rho c^2$, we have

$$\partial_\alpha \Pi = \left(v + \frac{c}{\gamma}\right)\partial_\alpha M, \quad \partial_\beta \Pi = \left(v - \frac{c}{\gamma}\right)\partial_\beta M \qquad (3.13a, b)$$

(we recall that the β-derivative is obtained by changing c into $-c$). In the same way we find, for the differentials of E,

$$\partial_\alpha E = \left[\frac{v^2}{2} + \frac{vc}{\gamma} + \frac{c^2}{\gamma(\gamma-1)}\right]\partial_\alpha M, \quad \partial_\beta E = \left[\frac{v^2}{2} - \frac{vc}{\gamma} + \frac{c^2}{\gamma(\gamma-1)}\right]\partial_\beta M. \qquad (3.14a, b)$$

Having established the above general results (valid for arbitrary entropy distribution), we now address the fundamental question of whether new conservation laws arise as a consequence of the existence of the symmetry. By this we mean the following: since $(\overline{\mathrm{T}})$ is an invariance transformation of the Euler equations, and the conserved quantities Π and E are known to exist, the transformed quantities $\overline{\Pi}$, $\overline{E}$ also exist, i.e. their Cauchy integrability conditions are satisfied. Then $\overline{\Pi}$, $\overline{E}$ either are expressible in terms of Π, E and of the physical variables r, t, P, etc., or else are independent quantities. In the latter case $\overline{\Pi}$, $\overline{E}$ constitute new conservation laws, whereas in the former no new conservation laws arise as a result of the symmetry.

In the present case the answer is provided by the fundamental transformation formula

$$\bar{v} = Mv - \Pi, \qquad (3.15)$$

which may be proved as follows. According to (2.11), the definition of $\bar{v}$ reads

$$\bar{v} = \int_0^M M\,\mathrm{d}v - \dot{R}(t),$$

where $\ddot{R}(t) = -P_0(t)$; hence $Mv - \bar{v} = \int_0^M v\,\mathrm{d}M + \dot{R}(t)$. Therefore

$$\frac{\partial(Mv - \bar{v})}{\partial M} = v,$$

$$\frac{\mathrm{d}(Mv - \bar{v})}{\mathrm{d}t} = \int_0^M \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}M + \ddot{R}(t) = \int_0^M -\frac{\partial P}{\partial M}\mathrm{d}M + \ddot{R}(t) = P_0(t) - P + \ddot{R}(t) = -P.$$

Thus $Mv - \bar{v}$ and Π, having the same partial derivatives, can differ at most by a constant. The constant can always be assumed to be zero, since Π itself is only determined up to an additive constant.

Owing to the 'symmetrical' nature of the operator (i.e. $(\overline{\mathrm{T}})^2 = 1$) we deduce, applying (3.15) once again,

$$v = \overline{M}\bar{v} - \overline{\Pi}, \qquad (3.16)$$

and, since $\overline{M} = 1/M$,

$$\overline{\Pi} = -\Pi/M. \qquad (3.17)$$

It may similarly be shown that the energy $\overline{E}$ is related to E in a simple way:

$$\overline{E} = -E + \frac{\Pi^2}{2M}, \qquad (3.18)$$

or, in a manifestly symmetrical way,

$$E+\bar{E}+\tfrac{1}{2}\Pi\bar{\Pi} = 0. \tag{3.19}$$

Thus no new conservation laws are introduced by the symmetry ($\bar{\mathrm{T}}$); still, it is quite interesting and unexpected that such simple relations should hold between the energy, momentum and their transformed quantities, in spite of the non-local nature of the transformation formulae (2.11), which involve integrals.

In addition, it ought to be noticed that *the momentum-conservation law itself may be viewed as a consequence of the symmetry* ($\bar{\mathrm{T}}$); since the existence of Π is proved by the ($\bar{\mathrm{T}}$)-transformation formula (3.15) ($\Pi = Mv-\bar{v}$).

3.3. *A Bäcklund transformation of the monatomic gas-flow equations*

We have already discussed in an earlier work (Gaffet 1981) another symmetry of the Euler equations, which we denoted by (T*); it holds for 'monatomic' gas flow only, that is to say, when the polytropic index takes the value

$$\gamma = \gamma_{\mathrm{mon}} \equiv (N+2)/N \tag{3.20}$$

in a space of dimension N. The basic transformation formulae read, for all N,

$$\left.\begin{aligned} t^* &= \frac{1}{t}, \quad M^* = -M, \\ r^* &= -\frac{r}{t}, \quad v^* = vt-r, \quad c^* = ct, \end{aligned}\right\} \tag{3.21}$$

together with

$$P^* = Pt^{N+2}, \quad \rho^* = \rho t^N,$$

where a star denotes transformed quantities. (T*) is a 'symmetry' too (i.e. $(\mathrm{T}^*)^2$ is the identity), and a Bäcklund transformation of the Euler equations. *It does yield a new conservation law*, that of energy E^*, which has the following current and density:

$$j_{\mathrm{L}}(E^*) = Pt(vt-r), \tag{3.22a}$$

$$\rho_{\mathrm{L}}(E^*) = \tfrac{1}{2}\{(v^2+\tfrac{1}{3}c^2)\,t^2-2rvt+r^2\}. \tag{3.22b}$$

The conservation law reads, in standard Eulerian formalism,

$$\frac{\partial}{\partial r}\rho\{(v^2+c^2)\,vt^2-2(v^2+\tfrac{1}{3}c^2)\,rt+vr^2\}+\frac{\partial}{\partial t}\rho\{(v^2+\tfrac{1}{3}c^2)\,t^2-2vrt+r^2\} = 0, \tag{3.23}$$

which might be considered a rather cumbersome formula, although much more complicated ones do occur in mathematical physics (see e.g. Landau & Lifschitz 1971, p. 306 equation (101.6)). It ought to be remarked that the degree of simplicity with which a conservation law may be formulated does not constitute a measure of its usefulness: its importance lies in that it provides *an exact integral of the equations, in closed form*. In the present case, (3.23) enables one to calculate the integral $\int_A^B \rho_{\mathrm{L}}[E^*]\,\mathrm{d}M = E^*(B)-E^*(A)$ *exactly* when the time evolution of the relevant physical data at the boundaries A, B is given. In the same way, the energy-conservation law enables one to calculate $E(B,t)-E(A,t)$ at any time t, when the energy fluxes are given as functions of time at the boundaries A, B.

Finally, we note that the momentum Π^* may be expressed in terms of the position coordinate $\bar{r}$:

$$\Pi^* = \bar{r}+Mr-\Pi t, \tag{3.24}$$

as may be shown by comparing their differentials; this result comes from the fact that both Π^* and $\bar{r}$ are connected with the centre-of-mass motion (see (2.11)). Formally, it is a consequence of the commutativity of the two operators:

$$(\bar{\mathrm{T}})(\mathrm{T}^*) \equiv (\mathrm{T}^*)(\bar{\mathrm{T}}). \tag{3.25}$$

4. General solutions

We recall that the pressure of a non-relativistic gas in a space of dimension N is related to the energy density *of translational motion* u_t by $P = (2/N)\,u_t$, and to the *total* energy density u by the polytropic law $P = (\gamma-1)\,u$, where γ is the adiabatic index; we thus have, for a monatomic gas,

$$\gamma_{\mathrm{mon}} = (N+2)/N. \tag{4.1}$$

In a one-dimensional space, which is the case we consider in the present paper, the monatomic index is thus $\gamma = 3$, although in three dimensions it takes the more familiar value $\gamma = \frac{5}{3}$. For a molecular gas with n rotational degrees of freedom, the equipartition principle predicts for the ratio of energy densities the value $u_t/u = n/(n+N)$, so that

$$\gamma = 1+2/(n+N). \tag{4.2}$$

Thus the classical theory predicts an adiabatic index of the general form

$$\gamma = (p+2)/p, \tag{4.3}$$

where p is an integer – the total number of degrees of freedom of the gas molecule. As we shall see, the above values of γ are also those for which some particularly interesting analytical results are available (§4.2).

4.1. *The case of monatomic gas flow* ($\gamma = 3$)

We first consider the monatomic index ($\gamma = 3$), which yields the simplest and most symmetrical results; the general solution may be set in the following form, if the flow is assumed isentropic (see §5.1 and Landau & Lifshitz 1959):

$$(v+c)\,t-r = \phi(v+c), \quad (v-c)\,t-r = \psi(v-c), \tag{4.4a,b}$$

which determines – though implicitly – v and c versus r and t coordinates, in terms of two arbitrary functions ϕ and ψ. The Riemann invariants $I^{\pm}$, which have the general expression $I^{\pm} = v \pm 2c/(\gamma-1)$ (see §5.1), are here $v+c$ and $v-c$. Choosing as characteristic coordinates

$$\alpha = \tfrac{1}{2}(v+c), \quad \beta = \tfrac{1}{2}(v-c), \tag{4.5}$$

the general solution (4.4) reads – solving for r, t and introducing new functions f and g:

$$t = \frac{f'(\alpha)-g'(\beta)}{\alpha-\beta}, \quad r = \frac{2(\beta f'-\alpha g')}{\alpha-\beta}, \tag{4.6a,b}$$

where f' and g' denote the ordinary derivatives of $f(\alpha)$ and $g(\beta)$.

The mass M, momentum Π and energy E may be obtained by quadrature from (3.4), (3.13) and (3.14) respectively; each integration can be performed in closed form, and we find

$$M = \Psi = 2(f-g)+(\beta-\alpha)(f'+g'), \tag{4.7}$$

$$\tfrac{1}{4}\Pi = (\alpha f-F)-(\beta g-G)+\tfrac{1}{6}(\beta-\alpha)\,[(2\alpha+\beta)f'+(2\beta+\alpha)\,g'], \tag{4.8}$$

where F and G are the primitives $\int f\,d\alpha$ and $\int g\,d\beta$ respectively.

The derivation of the general solution for the case of an entropy index $b = 3\gamma - 1$ (see (2.16)) is now straightforward, as it is connected to the isentropic solution by the symmetry ($\bar{\text{T}}$). From the transformation formulae (2.11) we have

$$t = -\frac{f' - g'}{\alpha - \beta}, \tag{4.9a}$$

$$M = \Psi^{-\frac{1}{3}} = \frac{1}{2(f-g) + (\beta - \alpha)(f' + g')}, \tag{4.9b}$$

$$c = (\alpha - \beta)[2(f-g) + (\beta - \alpha)(f' + g')] \tag{4.9c}$$

without any calculation. The velocity v may be found by quadrature from the second group of characteristic equations (3.3), but we may obtain it without integration through the transformation formula (3.15) and the expression (4.8) for the momentum:

$$v = 4(F-G) + 2(\beta - \alpha)(f+g) + \tfrac{1}{3}(\beta - \alpha)^2 (f' - g'). \tag{4.10}$$

The position coordinate r, which is determined by the first group of equations ((3.2)), may also be deduced without integration through the use of (3.24), where Π^* itself is deducible from expression (4.8) by a method developed in §5.1; we thus find

$$r = \frac{4(F-G)(f' - g')}{\beta - \alpha} - 4(F^* - G^*) + 2(f' - g')(\alpha f' + \beta g') + \tfrac{2}{3}(\beta - \alpha)(f' - g')^2, \tag{4.11}$$

where F^* and G^* are defined as follows:

$$F^*(\alpha) \equiv \frac{1}{2}\left(\int f'^2 \, \mathrm{d}\alpha + \alpha f'^2\right) - ff' \tag{4.12a}$$

$$G^*(\beta) \equiv \frac{1}{2}\left(\int g'^2 \, \mathrm{d}\beta + \beta g'^2\right) - gg'. \tag{4.12b}$$

The occurrence of the integrals $\int f'^2 \, \mathrm{d}\alpha$ and $\int g'^2 \, \mathrm{d}\beta$ will be interpreted in §5.1.

4.2. *A new symmetry connecting the flows of two different fluids* ($\gamma \neq \gamma'$)

We now present an explicit transformation (denoted by (T_+)) that has the remarkable property of connecting a flow characterized by an adiabatic index γ to the flow *of another fluid*, of index $\gamma' = 2 - 1/\gamma$; the transformation applies to the case of entropy distributions of the form (2.16), which we are considering in the present section. Comparing with (4.3), we note that that transformation is also characterized by the relation

$$p' = p + 2, \tag{4.13}$$

showing that (T_+) effectively *adds on two degrees of freedom* to the gas molecules. We do not elaborate about the meaning – if any – of that property, and proceed to give the mathematical definition and properties of (T_+).

The transformation is defined through the formulae

$$t' = \frac{1}{P}, \quad M' = \frac{1}{v}, \quad c' = Mvc, \tag{4.14}$$

and relates a flow of polytropic index γ, entropy index $b = 3\gamma - 1$, to the flow of another fluid with indices $\gamma' = (2\gamma - 1)/\gamma$, $b' = 3\gamma' - 1$. It is remarkable that the resulting transformation law for the velocity field *may be obtained in closed form too*; it reads

$$v' = \gamma\left\{\frac{Mv^2}{2} - E - \frac{Mc^2}{(\gamma-1)(2\gamma-1)}\right\}, \tag{4.15}$$

where E is the energy, defined up to an additive constant by its differentials ((3.14)). The transformed pressure is then

$$P' = \frac{-Mc^2P}{\gamma\gamma' v}. \tag{4.16}$$

In order to show that the functions P', v' of variables M', t' really satisfy the Euler equations for a fluid of index γ', we first observe that the partial derivatives $\partial/\partial M'$, $\partial/\partial t'$, coincide with: $-v^2(\partial/\partial v)|_P$, $-P^2(\partial/\partial P)|_v$ respectively; so that our first task is to express these derivatives in more standard form. We find†

$$\frac{\partial}{\partial P} = \frac{1}{\Delta}\left[\frac{\partial v}{\partial t}\frac{\partial}{\partial M} - \frac{\partial v}{\partial M}\frac{\partial}{\partial t}\right], \quad \frac{\partial}{\partial v} = \frac{1}{\Delta}\left[\frac{\partial P}{\partial M}\frac{\partial}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial}{\partial M}\right], \tag{4.17a, b}$$

where the Jacobian Δ is

$$\Delta = \frac{\partial P}{\partial M}\frac{\partial v}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial v}{\partial M}. \tag{4.18}$$

Substituting v', P', etc. from the transformation formulae (4.14)–(4.16), we find that the continuity equation

$$\frac{\partial v'}{\partial M'} + \frac{c'^2}{\gamma'^2 P'^2}\frac{\partial P'}{\partial t'} = 0$$

goes over into the Euler equation

$$\frac{\partial v}{\partial t} + \frac{\partial P}{\partial M} = 0,$$

and similarly that the transformed Euler equation reduces to the continuity equation for the original fluid. Finally one can easily check from (4.14)–(4.16) that the adiabatic equation of state is satisfied too, with the new choice of indices $\gamma' = (2\gamma-1)/\gamma$, $b' = 3\gamma'-1$. That completes the proof that the transformed flow is physically realizable, and obeys the Euler equations of a fluid with index γ', different from γ. The question of the fitting of the boundary conditions will be discussed in §4.4.

Let us mention finally that the momentum Π', which is determined through its differentials ((3.13)), is obtainable in closed form too:

$$\Pi' = -\frac{\gamma}{v}\left\{E - \Pi v + \frac{Mv^2}{2} + \frac{Mc^2}{(\gamma-1)(2\gamma-1)}\right\}. \tag{4.19}$$

4.3. *Case of a polytrope* $\gamma = \frac{5}{3}$

Starting from the general solution derived in §4.1 for the case $\gamma = 3$ ($b = 8$), the transformation formulae derived in §4.2 at once yield the solution corresponding to the new choice of indices $\gamma = \frac{5}{3}$ ($b = 4$). It reads

$$\left.\begin{aligned} t &= \frac{2(f-g)+(\beta-\alpha)(f'+g')}{(\beta-\alpha)^3}, \\ M &= \frac{1}{V}, \quad \Psi = -\tfrac{1}{5}V^5, \quad c = (\alpha-\beta)\,V, \end{aligned}\right\} \tag{4.20}$$

† Hereinafter the symbol $\partial/\partial t$ indicates the partial derivative at constant M, denoted by $\mathrm{d}/\mathrm{d}t$ in the preceding sections.

with

$$V \equiv 12(F-G)+6(\beta-\alpha)(f+g)+(\beta-\alpha)^2(f'-g').$$

The velocity v is given by

$$\tfrac{5}{3}v = 120(\Phi-\Gamma)+60(\beta-\alpha)(F+G)+12(\beta-\alpha)^2(f-g)+(\beta-\alpha)^3(f'+g'), \quad (4.21)$$

where we have introduced, in addition to the primitives F and G,

$$\Phi = \int F\,\mathrm{d}\alpha, \quad \Gamma = \int G\,\mathrm{d}\beta.$$

The momentum Π, as given by (4.19), reads

$$-\tfrac{5}{6}V\Pi = 60[(\alpha F-\Phi)-(\beta G-\Gamma)]+3(\beta-\alpha)[(7\alpha+3\beta)f+(3\alpha+7\beta)g] + (\beta-\alpha)^2[(3\alpha+2\beta)f'-(2\alpha+3\beta)g']. \quad (4.22)$$

Further application of the operator (T_+) yields the general solutions corresponding to indices $\gamma = \frac{7}{5}, \frac{9}{7}$, etc. (with entropy index $b = 3\gamma-1$).

In conclusion, we should stress the remarkable fact that (T_+) does not establish a correspondence between particles, i.e. between mass elements, of the two fluids since *M' is not* a function of M only ((4.14)). In other words, the Lagrangian coordinate *M does not remain Lagrangian* by that transformation.

4.4. *Practical applications*

Before discussing a few concrete applications of the present results, the following two general remarks are in order. The first is that in this work we deal with general symmetries of the Euler equations, with arbitrary entropy distribution (with the exception of the transformation (T_+), considered in §4.2, which is only valid for a special class of entropy profiles). These symmetries are used for deriving some quite general results concerning the form of the solution (§5). In addition, we obtain the *general solution* of the Euler equations in closed form, for the special class of entropy distributions. As usual with partial differential systems of the second order, the general solution involves two arbitrary functions (here denoted $f(\alpha)$ and $g(\beta)$), and any combination of initial or boundary conditions can be met, through an appropriate choice of the two functions; thus the problem of determining the solution to any concrete, particular problem is reduced to that of determining two functions of a single variable, a much simpler numerical task than the original one: to determine a function of *two* variables, e.g. $\rho(r,t)$.

It is, however, interesting to see whether the solution of a given concrete problem transforms, under the symmetries (T^*), $(\bar{\mathrm{T}})$ or (T_+), into another solution of practical interest; we shall address that question later in the present subsection.

This brings us to the second remark, which is that our main purpose in studying symmetries is not just to relate two or several different solutions by means of the symmetry; it is just that *no fundamental understanding of the equations can be claimed before all their symmetries are well understood*. Thus, the reason why e.g. Lorentz transformations are important is not so much that they can be used to generate new solutions; it is merely that they are important in their own right; they bring about covariance and analytical simplicity. That having been said, we now proceed to discuss some practical applications of our results.

4.4.1. *Applications of the transformation* (T^*)

One of the main results is that (T^*) brings about a new conservation law: that of E^*. Its practical importance was discussed in §3.3, together with its definition (3.23).

Broadly speaking, (T*) seems particularly useful for dealing with the problems of gasdynamics *in an expanding background*; that is to say, boundary conditions that describe a stationary background are transformed into the boundary conditions that are relevant for an expanding background. Thus (recalling that (T*) applies in any number of dimensions), interesting applications occur in astrophysical contexts such as multiple supernova explosions, or galactic explosions on a Hubble-flow cosmological background, a problem considered by Schwarz, Ostriker & Yahil (1975); the latter problem has been discussed by means of the transformation (T*) in Gaffet (1981).

4.4.2. *Applications of the transformation* ($\bar{\mathrm{T}}$)

The existence of a general solution in closed form for the entropy distribution $P/\rho^{\gamma} \approx 1/M^{3\gamma-1}$ ((2.16)) may find practical applications. In particular, problems involving strong shock propagation frequently result in a power-law deposition of entropy of the above form, although the power-law index need not coincide with that in (2.16); if it does, the subsequent evolution of the shock-heated gas will be described by the general solutions presented in §4.

As a specialization of the above equation (2.16), the general solution for the case of *a slowly varying entropy distribution* $P/\rho^{\gamma} = \sigma_0+\epsilon M$ (σ_0 = constant, $\epsilon \to 0$) is at once derived.

A more interesting type of application results from the following circumstances: I show in a forthcoming paper (Gaffet 1982) the existence of *an infinite number of conservation laws* when the equation of state assumes the form

$$P \approx \rho^3/M^4.$$

Then, by application of ($\bar{\mathrm{T}}$), one immediately generalizes the result to all entropy distributions of the general form

$$P \approx \frac{\rho^3}{(a_0+a_1 M+a_2 M^2)^4} \quad (a_0, a_1, a_2 \text{ are arbitrary constants}).$$

The above application illustrates the fact that the usefulness of a symmetry is not restricted to cases where the transformed boundary conditions are of any particular type.

Let us finally point out that the transformed boundary conditions will not be simple in the case of ($\bar{\mathrm{T}}$), if they involve the position coordinate r or even the velocity v (see the transformation formulae (2.11)); still, some degree of stability in the boundary conditions remains, at least in some cases: thus we note that the well-known energy-conserving Sedov–Taylor solutions (Sedov 1959) are transformed into other Sedov–Taylor solutions (with a different power-law index of the ambient density distribution).

5. The general solution, through the construction of generalized Riemann invariants

It was discovered by Riemann (1860) that the quantities

$$I^{\pm} = v \pm \frac{2c}{\gamma-1} \tag{5.1}$$

remain constant along ($C^{\pm}$) characteristics, in the case of isentropic flow. They are called Riemann invariants (hereinafter denoted by RI) and constitute a particular set of characteristic coordinates α, β. No generalization to the non-isentropic case has been known up to now, but we show in this section that a generalization does exist, at least when the entropy index takes the value $b = 3\gamma - 1$.

5.1. *Symmetrical nature of the general solution, expressed in terms of two pairs of Riemann invariants*

The symmetries (T^*, $\overline{\mathrm{T}}$, T_+, etc.) that we have considered in the present work all share in common with the Bäcklund transformations (Forsyth 1959) the fundamental property that the set of characteristic curves remains globally invariant; which implies that, given any RI pair $I^{\pm}$, the new quantities

$$K^{\pm} = \mathrm{T}^*[I^{\pm}] \tag{5.2}$$

are Riemann invariants too. We therefore have between $I^{\pm}$, $K^{\pm}$ the relations

$$K^+ = \phi(I^+), \quad K^- = \psi(I^-), \tag{5.3}$$

where the data of the two functions ϕ, ψ are arbitrary and serve to specify a given flow.

The above equations (5.3) constitute a formal solution of the Euler equations; the data of a single RI only (e.g. I^+) is needed in order to be able to write it down explicitly, since the other member of the pair (I^-) is obtainable by changing c to $-c$, as observed in §3.1.

The most striking application concerns the case of isentropic flow with $\gamma = 3$, where the RIs are: $I^{\pm} = v \pm c$, and the associated pair $K^{\pm} = (v \pm c)\,t - r$, according to our transformation formulae (3.21). The general solution therefore reads (see (5.3))

$$(v+c)\,t - r = \phi(v+c), \quad (v-c)\,t - r = \psi(v-c), \tag{5.4a, b}$$

which coincides with the result (4.4) already given in §4. It is remarkable that it can be derived *without any calculation, once the symmetry* (T^*) *is given*. We now discuss some algebraic consequences of the symmetry of the roles played by the pairs $I^{\pm}$, $K^{\pm}$, which reflect the symmetrical nature of (T^*) itself.

Let us denote for simplicity by (α, β), (α^*, β^*) the two RI pairs, and rewrite the solution (5.3) in the form

$$\alpha^* = f'(\alpha), \quad \beta^* = g'(\beta), \tag{5.5}$$

in order to have notation consistent with that of §4.1. Equation (4.6) thus reads

$$t = \frac{\alpha^* - \beta^*}{\alpha - \beta},$$

which clearly exhibits the essential property ($t^* = 1/t$) characterizing the symmetry (T^*). Our formulae of §4.1 also involve the primitive f of f', and higher-order primitives as well, such as $F = \int f\,\mathrm{d}\alpha$, $\Phi = \int F\,\mathrm{d}\alpha$. What are the transformed quantities f^*, F^*, Φ^*, ... ? The answer is, as $f = \int \alpha^*\,\mathrm{d}\alpha$,

$$f^* = \int \alpha\,\mathrm{d}\alpha^*,$$

or, integrating by parts,

$$f^* = \alpha f' - f. \tag{5.6}$$

In the same way, the quantity associated with $F(\alpha) \equiv \int f\,\mathrm{d}\alpha$ is $F^* = \int f^*\,\mathrm{d}\alpha^*$, and hence

$$F^* = \frac{1}{2}\int f'^2\,\mathrm{d}\alpha + \tfrac{1}{2}\alpha f'^2 - ff' \tag{5.7}$$

(F^* and G^* have already been introduced independently through (4.12)). Similar results hold for g^*, G^*, etc. versus the characteristic coordinate β.

Formula (5.7) accounts for the occurrence of the seemingly higher-order nonlinear terms $\int f'^2\,d\alpha$, $\int g'^2\,d\beta$ in the expression (4.11) for the position coordinate r; their presence is in fact necessary in order to make that expression manifestly symmetrical, that is to say, in order that

$$(\alpha^*-\beta^*)\,r^* \equiv -(\alpha-\beta)\,r,$$

as required by (3.21).

It should be noted that the above results (§5.1) are general and apply independently of the form of entropy distribution; reference to the results of §4.1, which hold in the isentropic case, was made for illustrative purposes only.

To summarize, we have shown the following: first, the data of one single RI is sufficient in order to obtain the solution (5.3); secondly, the most natural way of expressing the general solution is in terms of two pairs of variables $(\alpha, \beta; \alpha^*, \beta^*)$ which occur *in a completely symmetrical way*. The first property indicates the crucial importance of obtaining Riemann invariants in more general (i.e. non-isentropic) situations; the second property should, independently, constitute a powerful tool in the search for new cases of integrability.

5.2. *An extension of the Riemann invariants to non-isentropic flow*

In addition to the isentropic case, we can derive pairs of Riemann invariants for the case where $b = 3\gamma-1$ (for all γ), and thus obtain formal solutions of the form (5.3); we will treat here the case $\gamma = 3$ ($b = 8$).

The simplest RI-pair is obtained by application of operator $(\overline{\mathrm{T}})$ to the original Riemann invariants $v \pm c$, namely

$$I^{\pm} = \bar{v} \pm \bar{c} = M(v \pm c) - \Pi. \tag{5.8}$$

The associated pair is, by the symmetry (T*),

$$K^{\pm} = M[(v \pm c)\,t - r] + \Pi^*,$$

or, taking account of the identity (3.24),

$$K^{\pm} = I^{\pm} t + \bar{r}. \tag{5.9}$$

The above quantities have no explicit expression in terms of well-defined physical variables, and are only defined through their differentials. It is thus interesting to show that other invariants may be constructed, which are explicit. We propose the following solution:

$$L^{\pm} \equiv \frac{dK^{\pm}}{dI^{\pm}} = \frac{\partial K^{\pm}/\partial r}{\partial I^{\pm}/\partial r}, \tag{5.10}$$

which reads explicitly

$$L^{\pm} = t - \frac{1}{\dfrac{\partial(v \pm c)}{\partial r} \pm \dfrac{\rho c}{M}}. \tag{5.11}$$

The pair associated by the transformation (T*), however, does not constitute a new invariant; we have, indeed, from the definition (5.10)

$$\mathrm{T}^*[L^{\pm}] \equiv 1/L^{\pm}, \tag{5.12}$$

as $I^{\pm}$, $K^{\pm}$ are merely exchanged by the symmetry. It is still possible to obtain an independent pair which is explicit, e.g.

$$S^{\pm} \equiv \frac{dL^{\pm}}{dI^{\pm}}. \tag{5.13}$$

Let us remark that, in the notation developed in §5.1, the following identifications hold:

$$\left.\begin{aligned} I^+ &= \alpha, & K^+ &= f'(\alpha), \\ L^+ &= f''(\alpha), & S^+ &= f'''(\alpha), \end{aligned}\right\} \tag{5.14}$$

and similar identifications hold for the second members of each pair (I^-, K^-, etc.).

The invariants $L^{\pm}$, $S^{\pm}$ can be used for expressing the general solution in the form (5.3).

6. Conclusions

The present work constitutes a preliminary study of the symmetry properties of the one-dimensional Euler equations for adiabatic gas flow. Through considerations of scale invariance of the equations of classical mechanics, we have been led to discover two hidden symmetries, here denoted by ($\bar{\mathrm{T}}$), (T*), whose essential properties are that M goes over into $1/M$, and t into $1/t$ respectively,† where M is the Lagrangian mass coordinate and t the time coordinate; these two symmetries are susceptible to an interpretation at the microscopic level where the fluid is viewed as a many-particle system (see §2.2 and Gaffet 1981, §II.1).

The symmetry ($\bar{\mathrm{T}}$) affects the shape of the entropy distribution, a property that enabled us to derive new general solutions of the Euler equations starting from the already-known isentropic solutions. The solutions that we presented concern the cases where the entropy distribution is a power law of index $b = 3\gamma - 1$ (see (2.15) for the definition of b).

Even though the new solutions are restricted to that particular shape of entropy profile, it should be remembered that the domain of applicability of the two new symmetries themselves covers all functional forms of entropy distribution.

The fact that the symmetry (T*), on the other hand, does not affect the entropy distribution of the fluid makes it a Bäcklund transformation, from which integrability of the Euler equations might result. That possibility receives support from the existence of our new solutions, which shows that cases of complete integrability do indeed occur. In addition the new solutions, derived by means of transformation ($\bar{\mathrm{T}}$), show that – at least in some cases – the mathematics of isentropic and non-isentropic flow are equivalent. This certainly was an unexpected result.

We have also shown that, owing to the existence of the Bäcklund transformation (T*), the determination of a 'Riemann invariant' is a sufficient condition for integrability, and that the general solution assumes the form of (5.3). We show that Riemann invariants may be explicitly constructed even in non-isentropic cases, and at least in the case of an entropy profile of the form (2.16), to which our new solutions apply.

In a forthcoming paper, we intend to discuss the symmetries of the Euler equations more thoroughly, and in particular we introduce a third symmetry that relates the flow of two gases with different adiabatic index (γ, γ'). Like the first two, that symmetry applies independently of the form of entropy profile.

† More generally, M (respectively t) may be transformed into an arbitrary homographic function of M (or t) of the general type $M' = (a_0 M + a_1)/(b_0 M + b_1)$.

REFERENCES

COURANT, R. & FRIEDRICHS, K. 1948 *Supersonic Flow and Shock Waves*. Interscience.

FORSYTH, A. R. 1959 *Theory of Differential Equations*. Dover.

GAFFET, B. 1981 *Preprint RIFP*, 442 (May 1981).

GAFFET, B. 1982 An infinite Lie group of symmetry of one-dimensional gas flow, for a class of entropy distributions. *Preprint* (Sept. 1982).

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Addison-Wesley.

LANDAU, L. D. & LIFSHITZ, E. M. 1971 *The Classical Theory of Fields*, 3rd English edn. Pergamon.

RIEMANN, B. 1860 *Math. Phys. Klasse* **8**, 43.

SCHWARZ, J., OSTRIKER, J. P. & YAHIL, A. 1975 *Astrophys. J.* **202**, 1.

SEDOV, L. I. 1959 *Similarity and Dimensional Methods in Mechanics*. Academic.